

GEOMETRY OF CONFIGURATION SPACES OF TENSEGRITIES

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ABSTRACT. Consider a graph G with n vertices. In this paper we study geometric conditions for an n -tuple of points in \mathbb{R}^d to admit a non-zero self stress with underlying graph G . We introduce and investigate a natural stratification, depending on G , of the configuration space of all n -tuples in \mathbb{R}^d . In particular we find surgeries on graphs that give relations between different strata. Further we discuss questions related to geometric conditions defining the strata for plane tensegrities. We conclude the paper with particular examples of strata for tensegrities in the plane with a small number of vertices.

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1. INTRODUCTION

In his paper [16] J. C. Maxwell made one of the first approaches to the study of equilibrium states for frames under the action of static forces. He noted that the frames together with the forces give rise to reciprocal figures. In the second half of the twentieth century the artist K. Snelson built many surprising sculptures consisting of cables and bars that are actually such frames in equilibrium, see [22]. R. Buckminster Fuller introduced the name “tensegrity” for these constructions, combining the words “tension” and “integrity”. A nice overview of the history of tensegrity constructions is made by R. Motro in his book [17].

In mathematics, tensegrities were investigated in several papers. In [19] B. Roth and W. Whiteley and in [7] R. Connelly and W. Whiteley studied rigidity and flexibility of tensegrities, see also the survey about rigidity in [27].

In [26] N. L. White and W. Whiteley started the investigation of geometric realizability conditions for a tensegrity with prescribed bars and cables. In the preprint [12] M. de Guzmán describes other examples of geometric conditions for tensegrities.

Tensegrities have a wide range of applications in different branches of science and in architecture. For instance they are used in the study of viruses [5], cells [11], for construction of deployable mechanisms [21, 25], etc.

There are two main ways of working with tensegrities. The first deals with questions on rigidity of tensegrities, and the second with questions on the space of self-stresses on the edges of tensegrities. There is a certain duality between self-stresses and infinitesimal rigidity, more detailed explanations are given in Subsection 2.2. Nevertheless, there are many questions that are natural in the study of self-stresses and do not have nice reformulations in rigidity and vice versa. In this paper we investigate questions about self-stresses. Still we mention general links to rigidity.

Consider a frame F and the graph G corresponding to the frame F . We say that the graph G is *realizable as a tensegrity* on F if there exists an assignment of non-zero stresses to the edges of the frame such that the frame is in a static equilibrium.

Suppose a graph G is given. *Is the graph G realizable as a tensegrity for a frame representing G in general/special position? How many independent realizations does it have?* We develop a new technique to study such questions. We introduce special operations (*surgeries*) that change the graph in a certain way but preserve realizability properties. These surgeries are much in the spirit of matroid theory.

Let n be the number of vertices of G . Consider the configuration space of all n -tuples of points in \mathbb{R}^d . We define a natural stratification of the configuration space such that each stratum corresponds to a certain set of admissible tensegrities associated to G . Suppose

that one wants to obtain a construction with some edges of G replaced by struts and the others by cables, then he/she should take a configuration in a specific stratum of the stratification.

In this paper we prove that all the strata are semialgebraic sets, and therefore a notion of dimension is well-defined for them. This allows to ask the second question above more formally: *what is the minimal codimension of the strata in the configuration space that contains n -tuples of points admitting a tensegrity with underlying graph G ?* Our technique of surgeries on graphs gives the first answers in this case. In particular we obtain the list of all 6, 7, and 8 vertex tensegrities in the plane that are realizable for codimension 1 strata. We note that the complete answers to the above questions are not known to the authors.

N. L. White and W. Whiteley [26] and M. de Guzmán and D. Orden [13, 14] have found the geometric conditions of realizability of plane tensegrities with 6 vertices and of some other particular cases. We continue the investigation for other graphs (see Subsection 6.2). In all the observed examples the strata are defined by certain systems of geometric conditions. It turns out that all these geometric conditions are obtained from elementary ones:

- *two points coincide;*
- *three points are on a line;*
- *five points a, b, c, d, e satisfy: e is the intersection point of the lines passing through points a and b and points c and d respectively.* As W. Whiteley pointed out to us, these conditions are sufficient to describe any stratum for planar tensegrities (see Section 5).

This paper is organized as follows. We start in Section 2 with general definitions. In Subsection 2.1 we describe the configuration space of tensegrities associated to a given graph as a fibration over the affine space of all frameworks. Further in Subsection 2.2 we discuss a duality between spaces of self stresses and spaces of infinitesimal motions via the rigidity matrix. We introduce a natural stratification on the space of all frameworks and prove that all strata are semialgebraic sets (hence the strata have well-defined dimensions) in Subsection 2.3. In Section 3 we study the dimension of solutions for graphs on general configurations of points in \mathbb{R}^d . Later in this section we calculate the dimensions in the simplest cases, and formulate general open questions. In Section 4 we study surgeries on graphs and frameworks that induce isomorphisms of the spaces of self stresses for the frameworks. We give general definitions related to systems of geometric conditions for plane tensegrities in Section 5. Any stratum is a dense subset of the solution of one of such systems. Finally in Section 6 we give particular examples of graphs and their strata for tensegrities in the plane. We study the dimension of the space of self stresses in Subsection 6.1 and give tables of geometric conditions for codimension 1 strata for graphs with 8 vertices and less in Subsection 6.2.

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2. GENERAL DEFINITIONS

2.1. Configuration spaces of tensegrities. Recall standard definitions of tensegrities. We take them mostly from [6]. See also [20] for a collection of open problems and a good bibliography. We define $G(P)$ without additional “cable-bar-strut” structure specification (this is more convenient for the aims of the present paper).

Definition 2.1. Fix a positive integer d . Let $G = (V, E)$ be an arbitrary graph without loops and multiple edges. Let it have n vertices.

- A *configuration* is a finite collection P of n labeled points (p_1, p_2, \dots, p_n) , where each point p_i (also called a *vertex*) is in a fixed Euclidean space \mathbb{R}^d .
- The pair G and P is called a *tensegrity framework* and it is denoted as $G(P)$.
- We say that a *load* or *force* F acting on a framework $G(P)$ in \mathbb{R}^d is an assignment of a vector f_i in \mathbb{R}^d to each vertex i of G .
- We say that a *stress* w for a framework $G(P)$ in \mathbb{R}^d is an assignment of a real number $w_{i,j} = w_{j,i}$ (we call it an *edge-stress*) to each edge $p_i p_j$ of G . An edge-stress is regarded as a tension or a compression in the edge $p_i p_j$. For simplicity reasons we put $w_{i,j} = 0$ if there is no edge between the corresponding vertices. We say that w *resolves* a load F if the following vector equation holds for each vertex i of G :

$$f_i + \sum_{\{j|j \neq i\}} w_{i,j}(p_j - p_i) = 0.$$

By $p_j - p_i$ we denote the vector from the point p_i to the point p_j .

- A stress w is called a *self stress* if, the following equilibrium condition is fulfilled at every vertex p_i :

$$\sum_{\{j|j \neq i\}} w_{i,j}(p_j - p_i) = 0.$$

Actually, this means that the stress resolves the zero force.

- A couple $(G(P), w)$ is called a *tensegrity* if w is a self stress for the framework $G(P)$.

Denote by $W(n)$ the linear space of dimension n^2 of all edge-stresses $w_{i,j}$. Consider a framework $G(P)$ and denote by $W(G, P)$ the subset of $W(n)$ of all possible self stresses for $G(P)$. By definition of self stress, the set $W(G, P)$ is a linear subspace of $W(n)$.

Definition 2.2. The *configuration space of tensegrities* corresponding to the graph G is the set

$$\Omega_d(G) := \{(G(P), w) \mid P \in (\mathbb{R}^d)^n, w \in W(G, P)\}.$$

The set $\{G(P) \mid P \in (\mathbb{R}^d)^n\}$ is said to be the *base of the configuration space*, we denote it by $B_d(G)$.

If we forget about the edges between the points in all the frameworks, then we get natural bijections between $\Omega_d(G)$ and a subset of $(\mathbb{R}^d)^n \times W(n)$ and between $B_d(G)$ and $(\mathbb{R}^d)^n$. Later on we actually identify the last two pairs of sets. The bijections induce natural topologies on $\Omega_d(G)$ and $B_d(G)$.

Let π be the natural projection of $\Omega_d(G)$ to the base $B_d(G)$. This defines the structure of a fibration. For a given framework $G(P)$ of the base we call the set $W(G, P)$ the *linear fiber* at the point P (or at the framework $G(P)$) of the configuration space.

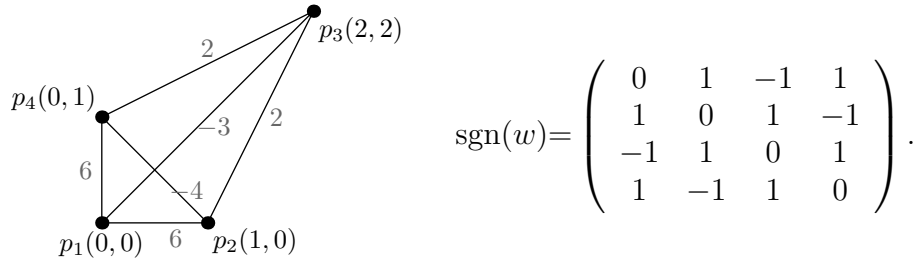
Consider a self stress w for the framework $G(P)$. We say that the edge $p_i p_j$ is a *cable* if $w_{i,j} < 0$ and a *strut* if $w_{i,j} > 0$. If we do not care about the sign of the edge-stress, we say that this edge is a *bar*.

Denote by “sgn” the sign function over \mathbb{R} .

Definition 2.3. Consider a framework $G(P)$ and one of its self stresses w . The $n \times n$ matrix $(\text{sgn}(w_{i,j}))$ is called the *strut-cable matrix* of the stress w and denoted by $\text{sgn}(w)$.

Let us give one example of a strut-cable matrix.

Example 2.4. Consider a configuration of four points in the plane: $p_1(0,0)$, $p_2(1,0)$, $p_3(2,2)$, $p_4(0,1)$ and a self stress w as on the picture: $w_{1,2} = 6$, $w_{1,3} = -3$, $w_{1,4} = 6$, $w_{2,3} = 2$, $w_{2,4} = -4$, $w_{3,4} = 2$. Then we have:



2.2. Rigidity matrix and infinitesimal rigidity. To study the dimension of the fibers $W(G, P)$ we use the rigidity matrix. Let us briefly recall its definition from [26]. Let $G(P)$ be a framework with vertices $V = \{p_1, \dots, p_n\}$ and edge set E . The *rigidity matrix* $M(G(P))$ of a framework $G(P)$ in dimension d is the $|E| \times dn$ matrix:

$$\begin{array}{l} \text{edge } \{a, b\} \\ \text{edge } \{a, f\} \\ \vdots \\ \text{edge } \{e, f\} \end{array} \begin{bmatrix} \begin{array}{c} \text{vertex } a \\ a_1 - b_1, \dots, a_d - b_d \end{array} & \begin{array}{c} \text{vertex } b \\ b_1 - a_1, \dots, b_d - a_d \end{array} & \cdots & \begin{array}{c} \text{vertex } f \\ 0, \dots, 0 \end{array} \\ \begin{array}{c} a_1 - f_1, \dots, a_d - f_d \\ 0, \dots, 0 \end{array} & \begin{array}{c} 0, \dots, 0 \\ 0, \dots, 0 \end{array} & \cdots & \begin{array}{c} f_1 - a_1, \dots, f_d - a_d \\ f_1 - e_1, \dots, f_d - e_d \end{array} \end{bmatrix}$$

So for the edge $\{e, f\}$ the matrix has the row with $e_1 - f_1, \dots, e_d - f_d$ in the columns of e and with $f_1 - e_1, \dots, f_d - e_d$ in the columns of f .

Let us replace all the edges of $G(P)$ by rigid bars with hinges at the vertices. The rigidity matrix gives information on the infinitesimal motions of such a configuration: an *infinitesimal motion* is defined as a solution of $M(G(P))X = 0$, where we regard X

as consisting of n vectors in \mathbb{R}^d (one vector at each vertex). Denote the space of all infinitesimal motions of $G(P)$ by $R(G, P)$. Then we have:

$$\dim R(G, P) = dn - \text{rank } M(G(P)).$$

We assume now that the affine space spanned by the vertices of the framework is the whole space \mathbb{R}^d . If $\dim R(G, P) = \frac{d(d+1)}{2}$ then there are no infinitesimal motions other than Euclidean isometries. In that case one calls the framework *infinitesimally rigid*.

It is interesting to note that the matrix transpose to the rigidity matrix plays a similar role for studying self stresses:

$$\dim W(G, P) = |E| - \text{rank } (M(G(P)))^t.$$

So we can write:

$$\dim W(G, P) - \dim R(G, P) = |E| - dn.$$

The most interesting case here is when $G(P)$ is infinitesimally rigid and when $|E|$ is minimal (i.e. equal to $nd - \frac{d(d+1)}{2}$). In the planary case such frameworks were studied by G. Laman.

Theorem 2.5. (G. Laman [15].) *Let $G = (V, E)$ be a graph satisfying $2|V| - |E| = 3$. Then a framework $G(P)$ on a generic point configuration P is infinitesimally rigid if and only if for every $X \subset V$ with $|X| \geq 2$, the subgraph induced by X has at most $2|X| - 3$ edges.*

A graph as in this theorem is called an *isostatic graph in \mathbb{R}^2* or a *Laman graph*. We remark here that in this context one calls a point configuration *generic* if there is no algebraic relation over the rational numbers between the coordinates of the points. One of the goals of the present paper is to give a more precise meaning to the word “generic” (see Remark 2.8). Laman’s theorem immediately implies the following.

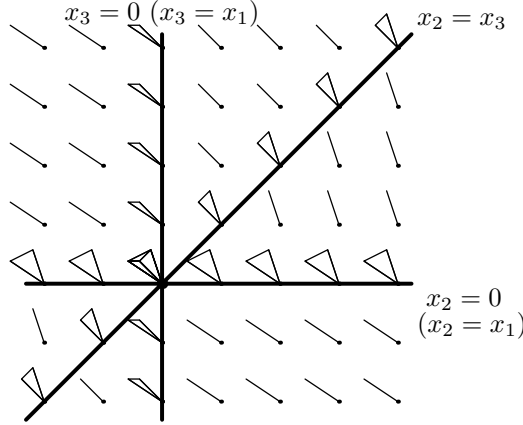
Corollary 2.6. *Let $G = (V, E)$ be a graph satisfying $2|V| - |E| = 3$. Then a framework $G(P)$ on a generic point configuration P has a nonzero self stress if and only if G is not isostatic in \mathbb{R}^2 .*

For additional information about the connection between infinitesimal rigidity and the existence of non-zero self stresses we refer to the papers [19] by B. Roth, W. Whiteley and [7] by R. Connelly and W. Whiteley.

2.3. Stratification of the base of a configuration space of tensegrities. Suppose we have some framework $G(P)$ and we want to find a cable-strut construction on it. Then *which edges can be replaced by cables, and which by struts? What is the geometric position of points for which given edges may be replaced by cables and the others by struts?* These questions lead to the following definition.

Definition 2.7. A linear fiber $W(G, P_1)$ is said to be *equivalent* to a linear fiber $W(G, P_2)$ if there exists a homeomorphism ξ between $W(G, P_1)$ and $W(G, P_2)$, such that for any self stress w in $W(G, P_1)$ the self stress $\xi(w)$ satisfies

$$\text{sgn}(\xi(w)) = \text{sgn}(w).$$


 FIGURE 1. The section $x_1 = 0$ of the stratification of $B_1(K_3)$.

The described equivalence relation gives us a stratification of the base $B_d(G) = (\mathbb{R}^d)^n$. A *stratum* is by definition a maximal connected set of points with equivalent linear fibers. Once we have proven Theorem 2.10, by general theory of semialgebraic sets (see for instance [3]) it follows that all strata are path-connected.

Remark 2.8. Using this definition, we can replace “generic point configuration” in Laman’s theorem (Theorem 2.5 and Corollary 2.6) by “point configuration in a stratum of codimension 0 in $B_2(G)$ ”.

Example 2.9. We describe the stratification of $B_1(K_3) = \mathbb{R}^3$ for the complete graph K_3 on three vertices, i.e., the case of 3-vertex tensegrities on a line. The point (x_1, x_2, x_3) in \mathbb{R}^3 corresponds to the framework with vertices $p_1 = (x_1)$, $p_2 = (x_2)$, and $p_3 = (x_3)$.

The stratification consists of 13 strata. There is 1 one-dimensional stratum, and there are 6 two-dimensional and 6 three-dimensional strata. The union of the strata of codimension greater than 1 is the union of three planes meeting in one line, the angles between these planes are all equivalent to $\pi/3$. On Figure 1 we show the plane section $x_1 = 0$ of the stratification. We indicate dimensions of corresponding fibers by segments, triangles, and a tetrahedron in the one-, two-, and three-dimensional cases respectively.

The one-dimensional stratum consists of frameworks with all vertices coinciding: $x_1 = x_2 = x_3$. The dimension of the fiber at a point of this stratum is three.

Any of the two-dimensional strata consists of frameworks with exactly two vertices coinciding. The strata are the connected components of the complement to the line $x_1 = x_2 = x_3$ in the union of the three planes $x_1 = x_2$, $x_1 = x_3$, and $x_2 = x_3$. The dimension of the fiber at a point of any of these strata is two.

Any of the three-dimensional strata consists of frameworks with distinct vertices. The strata are the connected components of the complement in \mathbb{R}^3 to the union of the three planes $x_1 = x_2$, $x_1 = x_3$, and $x_2 = x_3$. The dimension of the fiber at a point of any of these strata is one.

In general we have the following theorem.

Theorem 2.10. *Any stratum is a semialgebraic set.*

For the definition and basic properties of semialgebraic sets we refer the reader to [3].

We need two preliminary lemmas for the proof of the theorem, but first we introduce the following notation.

Let M be an arbitrary symmetric $n \times n$ -matrix with zeroes on the diagonal and all the other entries belonging to $\{-1, 0, 1\}$. Let i be an integer with $0 \leq i \leq n^2$. We say that a couple (M, i) is a *stratum symbol*.

For an arbitrary framework $G(P)$ we denote by $W_M(G, P)$ the set of all self stresses with strut-cable matrix $\text{sgn}(w)$ equal to M . The closure of $W_M(G, P)$ is a pointed polyhedral cone with vertex at the origin. The set $W_M(G, P)$ is homeomorphic to an open k -dimensional disc, we call k the dimension of $W_M(G, P)$ and denote it by $\dim(W_M(G, P))$.

For any stratum symbol (M, i) we denote by $\Xi(G, M, i)$ the set

$$\{(G(P), w) \mid w \in W(G, P), \text{sgn}(w) = M, \dim(W_M(G, P)) = i\} \subset \Omega_d(G).$$

Lemma 2.11. *For any stratum symbol (M, i) , the subset $\pi(\Xi(G, M, i))$ of the base $B_d(G)$ is either empty or it is a semialgebraic set.*

Proof. The set $\Xi(G, M, i)$ is a semialgebraic set since it is defined by a system of equations and inequalities in the coordinates of the vertices and the edge-stresses of the following three types:

- a) quadratic equilibrium condition equations;
- b) linear equations or inequalities specifying if the coordinate values $w_{i,j}$ are zeroes, positive, or negative reals;
- c) algebraic equations and inequalities defining respectively $\dim(W_M(G, P)) \leq i$ and $\dim(W_M(G, P)) \geq i$. Note that $\dim(W_M(G, P))$ is equal to the dimension of the linear space spanned by $W_M(G, P)$.

Let us make a small remark about item (c). At each framework we take the system of equilibrium conditions and equations of type $w_{i,j} = 0$ in the variables $w_{i,j}$. This system consists of the *equalities* of items (a) and (b). It is linear in the variables $w_{i,j}$. The corresponding matrix here is the rigidity matrix, its coefficients depend linearly on the coordinates of the framework vertices. The equations and inequalities of item (c) are defined by some determinants of submatrices of the rigidity matrix being equal or not equal to zero. Therefore, they are algebraic.

Since by the Tarski-Seidenberg theorem any projection of a semialgebraic set is semialgebraic, the set $\pi(\Xi(G, M, i))$ is semialgebraic. \square

Denote by $S(G, P)$ the set of all stratum symbols (M, i) that are realized by the point $G(P)$, in other words

$$S(G, P) = \{(M, i) \mid G(P) \in \pi(\Xi(G, M, i))\}.$$

Lemma 2.12. *Let $G(P_1)$ and $G(P_2)$ be two frameworks. Then $S(G, P_1) = S(G, P_2)$ if and only if the linear fiber $W(G, P_1)$ is equivalent to the linear fiber $W(G, P_2)$.*

Proof. Let the linear fiber at the point $G(P_1)$ be equivalent to the linear fiber at the point $G(P_2)$ then by definition we have

$$S(G, P_1) = S(G, P_2).$$

Suppose now that $S(G, P_1) = S(G, P_2)$. Let us denote by $\overline{W(G, P_i)}$ the one point compactification of the fiber $W(G, P_i)$ for $i = 1, 2$. So $\overline{W(G, P_i)}$ is homeomorphic to a sphere of dimension $\dim W(G, P_i)$.

For any point P and any M the set $W_M(G, P)$ is a convex cone homeomorphic to an open disc of dimension $\dim(W_M(G, P))$. So, for any point P we have a natural CW-decomposition of $\overline{W(G, P)}$ with cells $W_M(G, P)$ and the new one point cell.

A cell $W_{M'}(G, P_1)$ is adjacent to a cell $W_{M''}(G, P_1)$ iff the cell $W_{M'}(G, P_2)$ is adjacent to the cell $W_{M''}(G, P_2)$. This is true, since the couples of cells corresponding to M' and to M'' are defined by the same sets of equations and inequalities of type “ $>$ ”, and the closures of $W_{M'}(G, P_i)$ for $i = 1, 2$ are defined by the system defining $W_{M'}(G, P_i)$ with all “ $>$ ” in the inequalities replaced by “ \geq ”.

Therefore, there exists a homeomorphism of $\overline{W(G, P_1)}$ and $\overline{W(G, P_2)}$, sending all the cells $W_M(G, P_1)$ to the corresponding cells $W_M(G, P_2)$. We leave the proof of this statement as an exercise for the reader, this can be done by inductively constructing the homeomorphism on the k -skeletons of the CW-complexes.

Hence, the linear fibers $W(G, P_1)$ and $W(G, P_2)$ are equivalent. \square

Proof of Theorem 2.10. Let us prove the theorem for a stratum containing some point P . Consider any point P' in the stratum. By definition, $W(G, P)$ is equivalent to the space $W(G, P')$, and hence by Lemma 2.12, we have $S(G, P) = S(G, P')$.

Consider the following set

$$\bigcap_{(M, i) \in S(G, P)} \pi(\Xi(G, M, i)) \setminus \left(\bigcup_{(M, i) \notin S(G, P)} \pi(\Xi(G, M, i)) \right),$$

we denote it $\Sigma(P)$. So $\Sigma(P)$ is the set of frameworks $G(P')$ for which $S(G, P') = S(G, P)$. By Lemma 2.11 all the sets $\pi(\Xi(G, M, i))$ are semialgebraic. Therefore, the set $\Sigma(P)$ is semialgebraic. Denote by $\Sigma'(P)$ the connected component of $\Sigma(P)$ that contains the point P . Since the set $\Sigma(P)$ is semialgebraic, the set $\Sigma'(P)$ is also semialgebraic, see [3].

Let us show that $\Sigma'(P)$ is the stratum of $B_d(G)$ containing the point P . First, the set $\Sigma'(P)$ is contained in the stratum. This holds since $\Sigma'(P)$ is connected and consists of points with equivalent sets $S(G, P)$. And hence by Lemma 2.12 all the points of $\Sigma'(P)$ have equivalent linear fibers $W(G, P)$. Secondly, the stratum is contained in the space $\Sigma'(P)$. This holds since the stratum is connected and consists of points with equivalent linear fibers $W(G, P)$. Thus by Lemma 2.12 all the points of the stratum have equivalent sets $S(G, P)$.

As we have shown, the stratum containing P coincides with $\Sigma'(P)$ and hence it is semialgebraic. \square

From the above proof it follows that the total number of strata is finite.

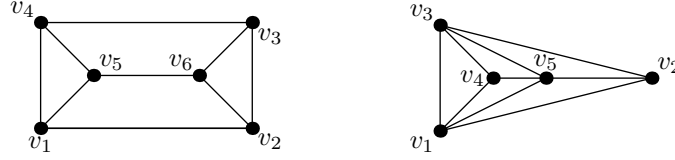


FIGURE 2. A graph with zero tensegrity 2-characteristic (the left one) and a graph whose 2-characteristic equals 2 (the right one).

3. ON THE TENSEGRITY d -CHARACTERISTIC OF GRAPHS

In this section we study the dimension of the linear fiber for graphs on a general point configuration in \mathbb{R}^d . We give a natural definition of the tensegrity d -characteristic of a graph and calculate it for the simplest graphs. We also mention some questions related to connections with rigidity matroids. In addition we formulate general open questions for further investigation.

3.1. Definition and basic properties of the tensegrity d -characteristic. Note that for any two points P_1 and P_2 of the same stratum S of the space $B_d(G)$ for a graph G we have

$$\dim(W(G, P_1)) = \dim(W(G, P_2)).$$

Denote this number by $\dim(G, S)$. Denote also by $\text{codim}(S)$ the integer

$$\dim(B_d(G)) - \dim(S).$$

Consider a graph G with at least one edge. We call the integer

$$\min\{\text{codim } S \mid S \text{ is a stratum of } B_d(G), \dim(G, S) > 0\}$$

the *codimension* of G and denote it by $\text{codim}_d(G)$.

Definition 3.1. We call the integer

$$\begin{cases} 1 - \text{codim}_d(G), & \text{if } \text{codim}_d(G) > 0 \\ \max\{\dim(W(G, P)) \mid G(P) \text{ contained in a codimension zero stratum}\}, & \text{otherwise} \end{cases}$$

the *tensegrity d -characteristic* of the graph G (or the d -TC of G for short), and denote it by $\tau_d(G)$.

Example 3.2. Consider the two graphs shown on Figure 2. The left one is a graph of codimension 1 in the plane, it can be realized as a tensegrity iff either the two triangles are in perspective position or the points of one of the two triples (v_1, v_4, v_5) or (v_2, v_3, v_6) lie on a line (for more details see [14]), so its 2-TC is zero. The graph on the right has a twodimensional space of self stresses for a general position plane framework, and hence its 2-TC equals two (we show this later in Proposition-Example 6.1). Notice that Corollary 2.6 applies as well to this example.

Proposition 3.3. *Let S_1 and S_2 be two strata of codimension 0. Let $G(P_1)$ and $G(P_2)$ be two points of the strata S_1 and S_2 respectively. Then the following holds:*

$$\dim(W(G, P_1)) = \dim(W(G, P_2)).$$

Proof. The equilibrium conditions give a linear system of equations in the variables $w_{i,j}$, at each framework linearly depending on the coordinates of the vertices. The dimension of the solution space is determined by the rank of the matrix of this system. The subset of $B_d(G)$ where the rank is not maximal is an algebraic subset of positive codimension. By definition, this set does not have elements in the strata S_1 and S_2 . This yields the statement of the proposition. \square

Corollary 3.4. *Let G be a graph. If $\tau_d(G) \geq 0$ then for every framework $G(P)$ in a codimension 0 stratum we have $\dim W(G, P) = \tau_d(G)$.* \square

3.2. Atoms and atom decomposition. In this subsection we recall a definition and some results of M. de Guzmán and D. Orden [14] that we use later.

Consider a point configuration P of $d+2$ points in general position in \mathbb{R}^d . Throughout this subsection ‘general position’ means that no $d+1$ of them are contained in a hyperplane. An *atom* in \mathbb{R}^d is a tensegrity $(K_{d+2}(P), w)$, where K_{d+2} is the complete graph on $d+2$ vertices and where w is a nonzero self stress.

According to [14, Section 2] the linear fiber $W(K_{d+2}, P)$ is one-dimensional for P in general position, in particular this implies $\tau_d(K_{d+2}) = 1$. In addition the edge-stress on every edge in the atom is nonzero. A more general statement holds.

Lemma 3.5. [14, Lemma 2.2] *Let $G(P)$ be a framework on a point configuration P in \mathbb{R}^d in general position. Let $p \in P$. Given a nonzero self stress on $G(P)$, then either at least $d+1$ of the edges incident to p receive nonzero edge-stress, or all of them have zero edge-stresses.*

M. de Guzmán and D. Orden showed that one can consider atoms as the building blocks of tensegrity structures. First, we explain how to add tensegrities. Let $T = (G(P), w)$ and $T' = (G'(P'), w')$ be two tensegrities. We define $T + T'$ as follows. The framework of $T + T'$ is $G(P) \cup G'(P')$, we take the union of vertices and edges. The edge-stress on a common edge $p_i p_j = p'_k p'_l$ is defined as $w_{i,j} + w'_{k,l}$ and on an edge appearing exactly in one of the original frameworks we put the original edge-stress. It is easy to see that the defined stress is a self stress, so $T + T'$ is a tensegrity.

Theorem 3.6. [14, Theorem 3.2] *Every tensegrity $(G(P), w)$ with a general position point configuration P and $w_{i,j} \neq 0$ on all edges of G is a finite sum of atoms. This decomposition is not unique in general.*

3.3. Rigidity Matroids, Connelly’s conjecture, and atom decomposition. Let us say a few words about a statement that is a rigidity analogue of atom decomposition for plane tensegrities. For this we need several definitions.

The rigidity matroid was introduced for studying rigidity questions, and hence it is sometimes related to verifying the existence of non-zero self stresses, i.e. to determine whether $\tau_n > 0$ or ≤ 0 (see Subsection 2.2 and the references mentioned there).

We restrict ourselves to the geometric description of rigidity matroids in the planar case, the notations and definitions are taken mostly from [9].

For a graph $G = (V, E)$ and a subset X of the vertex set V we denote by $i_G(X)$ the number of edges in the subgraph of G induced by X .

Definition 3.7. Consider a graph $G = (V, E)$. Let F be a non-empty subset of E , U the set of vertices incident with F , and $H = (U, F)$ the subgraph of G induced by F . We say that F is *independent* if for any subset X of U having more than two points we have

$$i_H(X) \leq 2|X| - 3.$$

The empty set is also defined to be independent.

All other subsets of E are said to be *dependent*.

The *rigidity matroid* $\mathcal{M}(G) = (E, \mathcal{I})$ is defined on the edge set of G by

$$\mathcal{I} = \{F \subseteq E : F \text{ is independent in } G\}.$$

If every proper subset of E is independent and $|E| = 2|V| - 2$ then G is called a *generic circuit*.

Now we define two operations on graphs.

Let G be a graph and v_1v_2 an edge of G . An *extension* of G along v_1v_2 is obtained from G by subdividing the edge v_1v_2 by a new vertex v and adding a new edge vw connecting v to some vertex other than v_1 and v_2 .

For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with two designated edges v_1w_1 and v_2w_2 the *2-sum* of G_1 and G_2 along the edge pair v_1w_1 and v_2w_2 is the graph obtained from $G_1 - v_1w_1$ and $G_2 - v_2w_2$ by identifying v_1 with v_2 and w_1 with w_2 .

The following theorem is similar to atom decomposition.

Theorem 3.8. (A. Berg, T. Jordán [2]) *A graph $G = (V, E)$ is a generic circuit if and only if G is a connected graph obtained from disjoint copies of K_4 's by taking 2-sums and applying extensions.*

This theorem was conjectured by R. Connelly in the 1980's and was then proved by A. Berg and T. Jordán in [2]. For the application to rigidity we refer to Section 6 of that paper.

Remark 3.9. There is a difference between the sum of tensegrities that have a common edge and the 2-sum of graphs, since in a 2-sum we remove the common edge, while for the sum of tensegrities this edge may only be removed if its resulting edge-stress is zero. Therefore an atom decomposition works for graphs that are not generic circuits as well.

3.4. Calculation of tensegrity d -characteristic in the simplest cases. We start this subsection with the formulation of a problem, we do not know the complete solution of it.

Problem 1. Give a general formula for $\tau_d(G)$ in terms of the combinatorics of the graph.

Let us calculate the d -TC for a complete graph, this will give us the maximal value of the d -TC for fixed number of vertices n and dimension d .

Proposition 3.10. *For any positive integers n and d satisfying $n \geq d+2$, we have*

$$\tau_d(K_n) = \frac{(n-d-1)(n-d)}{2}.$$

Proof. We work by induction on n . For $n = d+2$ the d -TC equals 1, as mentioned above. For $n > d+2$ we choose any point configuration P on n points such that no $d+1$ of them lie in a hyperplane. Take $p \in P$. Any tensegrity $(K_n(P), w)$ can be decomposed as a sum of $n-d-1$ atoms with p as vertex and a tensegrity on $P \setminus \{p\}$ with underlying graph K_{n-1} . Indeed, we can use such atoms to cancel the given edge-stresses on $n-d-1$ edges at p . Then there are only d edges left, so by Lemma 3.5 the edge-stresses on these edges equal zero. We conclude by induction that

$$\tau_d(K_n) = \tau_d(K_{n-1}) + n - d - 1 = \frac{(n-d-1)(n-d)}{2}.$$

□

Now we show how the d -TC behaves when we remove an edge of the graph.

Proposition 3.11. *Let G be some graph satisfying $\tau_d(G) > 1$. Let a graph G' be obtained from the graph G by erasing one edge. Then*

$$\tau_d(G) - \tau_d(G') \in \{0, 1\}.$$

Proof. Erasing one edge is equivalent to adding a new linear equation $w_{i,j} = 0$ to the linear system defining the space $W(G, P)$ for the graph G (for any point P). This implies that the space of solutions coincides with $W(G, P)$ or it is a hyperplane in $W(G, P)$.

So, first, $\tau_d(G') \leq \tau_d(G)$.

Secondly, since $\tau_d(G) = \dim(W(G, P_0))$ for some framework P_0 of a codimension 0 stratum for G (and therefore it belongs to a codimension 0 stratum for G'), then

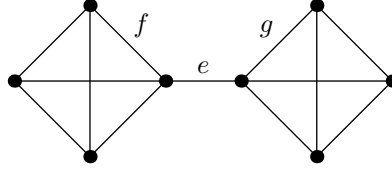
$$\tau_d(G') = \dim(W(G', P_0)) \geq \dim(W(G, P_0)) - 1 = \tau_d(G) - 1.$$

This completes the proof. □

As we show in the example below, erasing an edge does not always reduce the tensegrity characteristic.

Example 3.12. Consider the graph shown in Figure 3. Assume that this graph underlies a tensegrity. Then we can add an atom on the four leftmost vertices, to cancel the edge-stress on edge f for instance. This automatically cancels the edge-stresses on the edges connecting the four leftmost vertices by Lemma 3.5. We can do the same on the right. So the edge-stress on e is zero as well. Therefore the edge-stress on e was zero from the beginning and hence deleting e does not change the 2-TC. In Example 6.3 we give a less trivial example of this phenomenon.

Let us formulate two general corollaries of Proposition 3.11.

FIGURE 3. The edge-stress on the edge e is always zero.

Corollary 3.13. *Let G be a graph on n vertices and $m \in \mathbb{Z}_{>0}$. If G has*

$$m + \frac{n(n-1)}{2} - \tau_d(K_n) = m + dn - \frac{d^2 + d}{2}$$

edges, then $\tau_d(G) \geq m$.

Proof. Combine Proposition 3.10 and Proposition 3.11. □

The following corollary is useful for the calculation of the tensegrity d -characteristic. In Subsection 6.1 we use it to calculate all the tensegrity 2-characteristics for sufficiently connected graphs with less than 8 vertices.

Corollary 3.14. *Let G be a graph on n vertices with $\tau_d(G) \geq 0$. Assume that G has*

$$dn - \frac{d^2 + d}{2} + \tau_d(G)$$

edges. Then for any graph H that can be obtained from G by adding N edges we have

$$\tau_d(H) = \tau_d(G) + N.$$

Proof. We delete $\tau_d(K_n) - \tau_d(G) - N$ edges from K_n to reach H . If the d -TC does not drop by 1 at one of these steps, then we apply Proposition 3.11 an additional N times to H to reach G . This leads to a wrong value of $\tau_d(G)$. So the d -TC drops by one in each of the first $\tau_d(K_n) - \tau_d(G) - N$ steps and the formula for $\tau_d(H)$ follows. □

Example 3.15. A *pseudo-triangle* is a planar polygon with exactly three vertices at which the angles are less than π . Let G be a planar graph with n vertices and k edges that admits a pseudo-triangular embedding $G(P)$ in the plane, i.e. a non-crossing embedding such that the outer face is convex and all interior faces are pseudo-triangles. It is obvious that a pseudo-triangular embedding $G(P)$ belongs to a codimension 0 stratum of $B_2(G)$. By Lemma 2 of [18] we find that

- $\tau_2(G) = k - (2n - 3)$ if $k - (2n - 3) \geq 1$,
- $\tau_2(G) \leq 0$ if $k - (2n - 3) = 0$.

(Note that for pseudo-triangular embeddings we always have $k \geq 2n - 3$.)

4. SURGERIES ON GRAPHS THAT PRESERVE THE DIMENSION OF THE FIBERS

In this section we describe operations that one can perform on a graph without changing the dimensions of the corresponding fibers for the frameworks. We refer to such operations

as *surgeries*. The first type of surgeries is for general dimension, while the other two are restricted to dimension $d = 2$. We do not know other similar operations that are not compositions of the surgeries described below.

The idea of surgeries is analogous to the idea of Reidemeister moves in knot theory. If two graphs are connected by a sequence of surgeries, then one obtains tensegrities for the first graph from tensegrities for the second graph and vice versa.

We essentially use surgeries to calculate the list of geometric conditions for the strata for (sufficiently connected) graphs with less than 9 vertices and with zero 2-TC in Subsection 6.2.

4.1. General surgeries in arbitrary dimension. For an edge e of a graph G we denote by G_e the graph obtained from G by removing e .

Denote by $\Sigma_d(G)$ the union of codimension zero strata in $B_d(G)$. Let G be a graph and H a subgraph. Consider the map that takes a framework for G to the framework for H by forgetting all the vertices and edges of G that are not in H . Denote by $\Sigma_d(G, H)$ the preimage of $\Sigma_d(H)$ for this map.

Proposition 4.1. (Edge exchange) *Let G be a graph and H an induced subgraph with $\tau_d(H) = 1$. Consider a configuration P_0 lying in $\Sigma_d(G, H)$. Suppose that there exists a self stress on the framework $G(P_0)$ that has nonzero edge-stresses for all edges of H and zero edge-stresses on the other edges. Let e_1, e_2 be edges of H . Then for any $P \in \Sigma_d(G, H)$ we have*

$$W(G_{e_1}, P) \cong W(G_{e_2}, P).$$

The corresponding surgery takes the graph G_{e_1} to G_{e_2} , or vice versa.

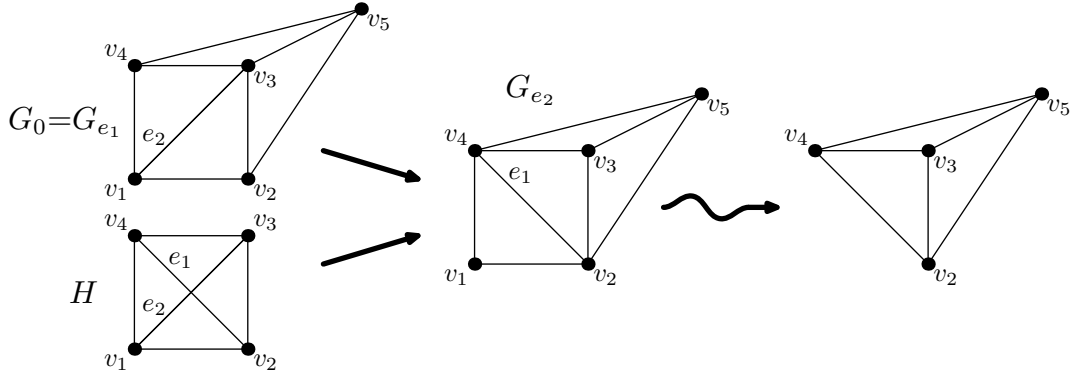
Remark 4.2. We always have the inclusion $\Sigma_d(G) \subset \Sigma_d(G, H)$, this follows directly from the definition of the strata. Nevertheless the set $\Sigma_d(G, H)$ usually contains many strata of $B_d(G)$ of positive codimension. So Proposition 4.1 is applicable to all strata of codimension zero as well as to some strata of positive codimension.

For the proof of Proposition 4.1 we need the following lemma.

Lemma 4.3. *Let G be a graph with $\tau_d(G) = 1$ and e one of its edges. Suppose that there exists a configuration $P_0 \in \Sigma_d(G)$ and a nonzero self stress w_0 such that $w_0(e) = 0$. Then for any tensegrity $(G(P), w)$ with $P \in \Sigma_d(G)$ we get $w(e) = 0$.*

Proof. Since $\tau_d(G) = 1$ and $P_0 \in \Sigma_d(G)$, any tensegrity $(G(P_0), w)$ satisfies the condition $w(e) = 0$. Therefore, any tensegrity with P in the same stratum as P_0 has zero edge-stress at e . So the condition *always to have zero edge-stress at e* defines a somewhere dense subset S in $B_d(G)$. Since the condition is defined by a solution of a certain linear system, S is dense in $B_d(G)$. It follows that $\Sigma_d(G)$ is a subset of S . \square

Proof of Proposition 4.1. From Lemma 4.3 we have that for any configuration of $\Sigma_d(H)$ there exist a unique up to a scalar self stress that is nonzero at each edge of H . The uniqueness follows from the fact that $\tau_d(H) = 1$. Hence for any configuration of $\Sigma_d(G, H)$

FIGURE 4. This shows that $\tau_2(G_0) = 1$.

there exists a unique up to a scalar self stress that is nonzero at each edge of H and zero at all other edges of G .

For any $P \in \Sigma_d(G, H)$ we obtain an isomorphism between $W(G_{e_1}, P)$ and $W(G_{e_2}, P)$ by adding the unique tensegrity on the underlying subgraph H of G that cancels the edge-stress on e_2 , considered as edge of G_{e_1} . \square

In particular one can use atoms (i.e. $H = K_{d+2}$) in the above proposition.

Corollary 4.4. *In the notation and with the conditions of Proposition 4.1 we have: if either $\tau_d(G_{e_1}) > 0$ or $\tau_d(G_{e_2}) > 0$ then*

$$\tau_d(G_{e_1}) = \tau_d(G_{e_2}).$$

Proof. The statement follows directly from Proposition 4.1 and Corollary 3.4. \square

Let us show how to use the above corollary to compute the tensegrity characteristic.

Example 4.5. We calculate the 2-TC of the graph G_0 shown in Figure 4. Consider the atom H on the vertices v_1, v_2, v_3 , and v_4 and let e_1, e_2 be the edges v_2v_4, v_1v_3 respectively. Denote by G the graph obtained from G_0 by adding the edge e_2 . So the graph G_0 is actually G_{e_1} . By Corollary 3.13 we have $\tau_2(G_0) \geq 1$, and hence it is possible to apply Corollary 4.4. Consider the graph G_{e_2} , it is shown in Figure 4 in the middle. The degree of the vertex v_1 in this graph equals 2, so by Lemma 3.5 the edge-stresses on its incoming edges equal zero if the points v_1, v_2 , and v_4 are not on a line. After removing these two edges and the vertex v_1 we get the graph of an atom. Therefore,

$$\tau_2(G_0) = 1.$$

4.2. Additional surgeries in dimension two. In this subsection we study two surgeries on edges of plane frameworks that do not change the dimension of the fibers of the frameworks.

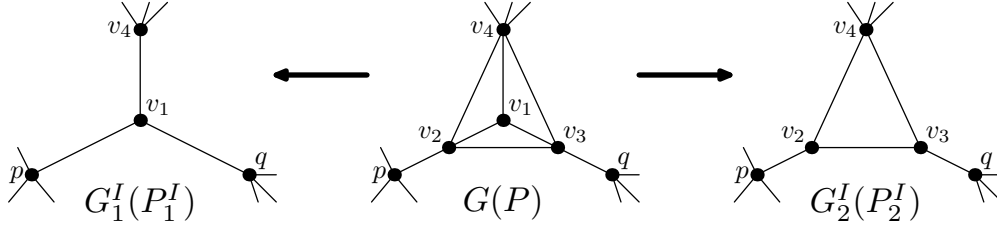


FIGURE 5. Surgery I.

Surgery I. Consider a graph G and a framework $G(P)$. Let G contain the complete graph K_4 with vertices v_1, v_2, v_3 , and v_4 as an induced subgraph. Suppose that the edges between v_1, v_2, v_3, v_4 and other vertices of G are as follows:

- pv_2 and qv_3 for unique vertices p and q ;
- the edges pv_1 and qv_1 ;
- any set of edges from v_4 .

In addition we require that the framework $G(P)$ has the triples of points (p, v_1, v_2) and (q, v_1, v_3) on one line. See Figure 5 in the middle.

Let us delete from the graph G the vertices v_2 and v_3 (the vertex v_1) with all edges adjacent to them. We denote the resulting graph by G_1^I (by G_2^I respectively). The corresponding framework is denoted by $G_1^I(P_1^I)$ (by $G_2^I(P_2^I)$ respectively). See Figure 5 on the left (on the right). Surgery I takes G_1^I to G_2^I or vice versa.

Remark 4.6. This surgery looks like the ΔY exchange in matroid theory, but it is not exactly the same. For instance, to go from $G_2^I(P_2^I)$ to $G_1^I(P_1^I)$ in Figure 5 we basically replace the triangle by a 3-star, but we assume here that the triangle has two vertices of degree 3.

Proposition 4.7. Consider the frameworks $G(P)$, $G_1^I(P_1^I)$, and $G_2^I(P_2^I)$ as above. If the triples of points (p, v_2, v_3) , (q, v_2, v_3) , (p, v_2, v_4) , (q, v_3, v_4) and (v_2, v_3, v_4) are not on a line then we have

$$W(G_1^I, P_1^I) \cong W(G_2^I, P_2^I).$$

Proof. We explain how to go from $W(G_2^I, P_2^I)$ to $W(G_1^I, P_1^I)$. The inverse map is simply given by the reverse construction. By the conditions the intersection point v_1 of pv_2 and qv_3 is uniquely defined and not on the lines through v_2 and v_4 or v_3 and v_4 . We add the uniquely defined atom on v_1, v_2, v_3, v_4 to $G_2^I(P_2^I)$ that cancels the edge-stress on v_2v_3 . Since p, v_2, v_1 lie on one line, this surgery also cancels the edge-stress on v_2v_4 and similarly for v_3v_4 . Due to the equilibrium condition at v_2 , we can replace the edges pv_2 and v_2v_1 with their edge-stresses $w_{p,2}$ and $w_{2,1}$ by an edge pv_1 with edge-stress $w_{p,1}$ defined by one of the following vector equations:

$$w_{p,2}(v_2 - p) = w_{p,1}(v_1 - p) = w_{2,1}(v_1 - v_2).$$

This uniquely defines a self stress on $G_1^I(P_1^I)$. □

Corollary 4.8. *Assume that one of the following conditions holds:*

- (1) $\tau_2(G_1^I) > 0$ or $\tau_2(G_2^I) > 0$.
- (2) $\tau_2(G_1^I) = 0$ and there is a codimension 1 stratum S of $B_2(G_1^I)$ such that
 - $\dim W(G_1^I, P) > 0$ for a $G_1^I(P)$ in the stratum S ,
 - the stratum S is not contained in the subset of $B_2(G_1^I)$ of frameworks having one of the triples of points (p, v_1, q) , (p, v_1, v_4) , or (q, v_1, v_4) on one line.
- (3) $\tau_2(G_2^I) = 0$ and there is a codimension 1 stratum S' of $B_2(G_2^I)$ such that
 - $\dim W(G_2^I, P') > 0$ for a $G_2^I(P')$ in the stratum S' ,
 - the stratum S' is not contained in the subset of $B_2(G_2^I)$ of frameworks having (p, v_2, v_3) , (q, v_2, v_3) , (p, v_2, v_4) , (q, v_3, v_4) , or (v_2, v_3, v_4) on one line.

Then

$$\tau_2(G_1^I) = \tau_2(G_2^I).$$

Proof. Let A be the subset of $B_2(G_2^I)$ of frameworks having (p, v_2, v_3) , (q, v_2, v_3) , (p, v_2, v_4) , (q, v_3, v_4) or (v_2, v_3, v_4) on one line. Let B be the subset of $B_2(G_1^I)$ of frameworks having (p, v_1, q) , (p, v_1, v_4) or (q, v_1, v_4) on one line. Note that A and B are of codimension 1. The proof of Proposition 4.7 gives a surjective map

$$\varphi : B_2(G_2^I) \setminus A \rightarrow B_2(G_1^I) \setminus B$$

inducing an isomorphism between the linear fibers above $G(P) \in B_2(G_2^I) \setminus A$ and $\varphi(G(P))$. Now in all the cases (1)—(3) the statement of the corollary follows directly from the definition of the tensegrity characteristic. \square

Surgery II. Consider a graph G and a framework $G(P)$. Let G contain the complete graph K_4 with vertices v_1, v_2, v_3 , and v_4 as an induced subgraph. Suppose that the set of edges between v_1, v_2, v_3, v_4 and other vertices of G is

$$\{pv_1, pv_2, qv_1, qv_3, rv_2, rv_4, sv_3, sv_4\},$$

for unique points p, q, r, s . In addition we require that the framework $G(P)$ has the triples of points

$$(p, v_1, v_2), \quad (q, v_1, v_3), \quad (r, v_2, v_4), \quad \text{and} \quad (s, v_3, v_4)$$

on one line. See Figure 6 in the middle.

Let us delete from the graph G the vertices v_1 and v_4 (v_2 and v_3) with all edges adjacent to them. We denote the resulting graph by G_1^{II} (by G_2^{II} respectively). The corresponding framework is denoted by $G_1^{II}(P_1^{II})$ (by $G_2^{II}(P_2^{II})$ respectively). See Figure 6 on the left (on the right). Surgery II takes G_1^{II} to G_2^{II} or vice versa.

The proofs of the proposition and corollary below are similar to the proofs of Proposition 4.7 and Corollary 4.8.

Proposition 4.9. *Consider the frameworks $G(P)$, $G_1^{II}(P_1^{II})$, and $G_2^{II}(P_2^{II})$ as above. If non of the triples of points (p, q, v_1) , (p, v_1, v_4) , (r, v_1, v_4) , (q, v_1, v_4) , (s, v_1, v_4) , or (r, s, v_4) lie on a line then we have*

$$W(G_1^{II}, P_1^{II}) \cong W(G_2^{II}, P_2^{II}).$$

\square

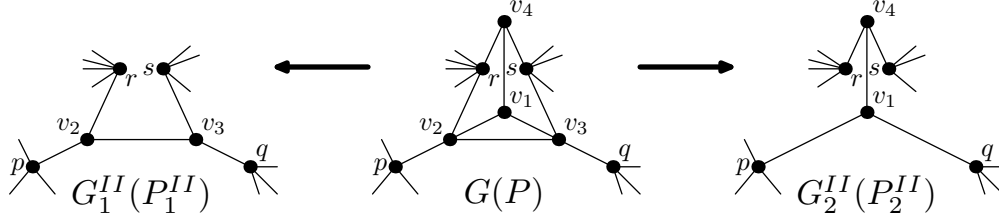


FIGURE 6. Surgery II.

Corollary 4.10. *Assume that one of the following conditions holds:*

- (1) $\tau_2(G_1^{II}) > 0$ or $\tau_2(G_2^{II}) > 0$.
- (2) $\tau_2(G_1^{II}) = 0$ and there is a codimension 1 stratum S of $B_2(G_1^{II})$ such that
 - $\dim W(G_1^{II}, P) > 0$ for a $G_1^{II}(P)$ in the stratum S ,
 - the stratum S is not contained in the subset of $B_2(G_1^{II})$ of frameworks having (p, v_2, v_3) , (q, v_2, v_3) , (p, v_2, r) , (q, v_3, s) , (r, v_2, v_3) , or (s, v_2, v_3) on one line.
- (3) $\tau_2(G_2^{II}) = 0$ and there is a codimension 1 stratum S' of $B_2(G_2^{II})$ such that
 - $\dim W(G_2^{II}, P') > 0$ for a $G_2^{II}(P')$ in the stratum S' ,
 - the stratum S' is not contained in the subset of $B_2(G_2^{II})$ of frameworks having (p, q, v_1) , (p, v_1, v_4) , (r, v_1, v_4) , (q, v_1, v_4) , (s, v_1, v_4) , or (r, s, v_4) on one line.

Then

$$\tau_2(G_1^{II}) = \tau_2(G_2^{II}).$$

□

5. GEOMETRIC RELATIONS FOR STRATA AND COMPLEXITY OF TENSEGRITIES IN TWO-DIMENSIONAL CASE

In all the observed examples of plane tensegrities with a given graph the strata for which a tensegrity is realizable are defined by certain geometric conditions on the points of the corresponding frameworks. In this section we study such geometric conditions. In Subsection 5.1 we describe an example of a geometric condition for a particular graph. Further, in Subsection 5.2 we give general definitions related to systems of geometric conditions. Finally, in Subsections 5.3 and 5.4 we state theorems and formulate open questions related to the geometric nature of tensegrity strata.

To avoid problems with describing annoying cases of parallel/nonparallel lines we extend the plane \mathbb{R}^2 to the projective space. It is convenient for us to consider the following model of the projective space: $\mathbb{R}P^2 = \mathbb{R}^2 \cup l_\infty$. The set of points l_∞ is the set of all “directions” in the plane. The set of lines of $\mathbb{R}P^2$ is the set of all plane lines (each plane line contains now a new point of l_∞ that is the direction of l) together with the line l_∞ . Now any two lines intersect at exactly one point.

5.1. A simple example. First, we study the graph shown in Figure 2 on the left, we denote it by G_0 . In [26] N. L. White and W. Whiteley proved that the 2-TC of this graph is zero. They showed that there exists a nonzero tensegrity with graph G_0 and framework

P iff the points of P satisfy one of the following three conditions:

- i) the lines v_1v_2 , v_3v_4 , and v_5v_6 have a common nonempty intersection (in $\mathbb{R}P^2$);
- ii) the vertices v_1 , v_4 , and v_5 are in one line;
- iii) the vertices v_2 , v_3 , and v_6 are in one line.

We remind that the base $B(G_0)$ of the configuration space is \mathbb{R}^{12} with coordinates $(x_1, y_1, \dots, x_6, y_6)$, where (x_i, y_i) are the coordinates of v_i . Condition (i) defines a degree 4 hypersurface with equation

$$\det \begin{pmatrix} y_1 - y_2 & y_3 - y_4 & y_5 - y_6 \\ x_2 - x_1 & x_4 - x_3 & x_6 - x_5 \\ x_1y_2 - x_2y_1 & x_3y_4 - x_4y_3 & x_5y_6 - x_6y_5 \end{pmatrix} = 0.$$

and Conditions (ii) and (iii) define the conics

$$\begin{aligned} x_1y_4 + x_4y_5 + x_5y_1 - x_1y_5 - x_4y_1 - x_5y_4 &= 0, \quad \text{and} \\ x_2y_3 + x_3y_6 + x_6y_2 - x_2y_6 - x_3y_2 - x_6y_3 &= 0 \end{aligned}$$

respectively.

5.2. Systems of geometric conditions. First we introduce two notations as in the Cayley algebra (for basic introduction to the Cayley algebra, also called Grassmann or double algebra, we refer to the works [1], [8], and [24]). Denote a line through the points p and q by $p \vee q$, or pq for short. For two lines l_1 and l_2 we denote by $l_1 \wedge l_2$ the intersection point of the lines.

Let us define the following three elementary geometrical conditions. Consider an ordered subset $P = \{p_1, \dots, p_n\}$ of the projective plane.

2-point condition. We say that the subset P satisfies the condition $p_i = p_j$ if p_i coincides with p_j .

3-point condition. We say that the subset P satisfies the condition

$$p_i \nabla p_j \nabla p_k = 0$$

if the points p_i , p_j , and p_k are on a line.

5-point condition. We say that the subset P satisfies the condition

$$p_i = [p_j, p_{j'}; p_k, p_{k'}]$$

if the four points $p_j, p_{j'}, p_k$, and $p_{k'}$ are on a line and p_i also belongs to this line, or if $p_i = (p_j \vee p_{j'}) \wedge (p_k \vee p_{k'})$.

Note that we give the last definition in terms of closures, since $pq \wedge rs$ is not defined for all 4-tuples, but for a dense subset.

Definition 5.1. Consider a system of elementary geometric conditions for ordered n -point subsets of $\mathbb{R}P^2$, and let $m \leq n$.

— We say that the ordered n -point subset P of projective plane satisfies the system of elementary geometric conditions if P satisfies each of these conditions.

— We say that the ordered subset $\{p_1, \dots, p_m\}$ *satisfies conditionally* the system of elementary geometric conditions if there exist points q_1, \dots, q_{n-m} such that the ordered set

$$\{p_1, \dots, p_m, q_1, \dots, q_{n-m}\}$$

satisfies the system. We call the number $n-m$ *the conditional number* of the system.

Example 5.2. The condition that *six points* p_1, \dots, p_6 *lie on a conic* is equivalent to the following geometric conditional system:

$$\begin{cases} q_1 = [p_1, p_2; p_4, p_5] \\ q_2 = [p_2, p_3; p_5, p_6] \\ q_3 = [p_3, p_4; p_1, p_6] \\ q_1 \nabla q_2 \nabla q_3 = 0 \end{cases}.$$

This is a reformulation of Pascal's theorem. The conditional number is 3 here.

We can rewrite the system as follows, for short:

$$(p_1 p_2 \wedge p_4 p_5) \nabla (p_2 p_3 \wedge p_5 p_6) \nabla (p_3 p_4 \wedge p_1 p_6) = 0.$$

Example 5.3. The condition for six points p_1, \dots, p_6 that *the lines* $p_1 p_2$, $p_3 p_4$, *and* $p_5 p_6$ *have a common point* is equivalent to the following geometric conditional system:

$$\begin{cases} q_1 = [p_1, p_2; p_3, p_4] \\ q_1 \nabla p_5 \nabla p_6 = 0 \end{cases},$$

or in a shorter form:

$$(p_1 p_2 \wedge p_3 p_4) \nabla p_5 \nabla p_6 = 0.$$

The conditional number of the system is 1.

5.3. Conjecture on geometric structure of the strata. For a given positive integer k and a graph G consider the set of all frameworks $G(P)$ at which the dimension of the fiber $W(G, P)$ is greater than or equal to k . We call this set the (G, k) -*stratum*. Since any (G, k) -stratum is a finite union of strata of the base $B_2(G)$, it is semialgebraic.

Definition 5.4. Let G be a graph and k be a positive integer. The (G, k) -stratum is said to be *geometric* if it is a finite union of the sets of conditional solutions of systems of geometric conditions (in these systems p_1, \dots, p_m correspond to the vertices of the graph).

Theorem 5.5. *For any graph and any integer k the (G, k) -stratum is geometric.*

This result was communicated to us by W. Whiteley.

Proof. The (G, k) -stratum is defined as the set of frameworks $G(P)$ in $B_2(G)$ where it holds $\dim W(G, P) \geq k$. As explained in Subsection 2.2, this corresponds to the locus where the rigidity matrix $M(G(P))$ has rank less than $|E| - k$, where E is the edge set of G . So the (G, k) -stratum is the set of frameworks where the sum of the squares of all $(|E| - k + 1)$ -minors of the rigidity matrix vanishes, and hence it is described by a polynomial f in the coordinates of the points of P . According to the proof of Theorem 5.10 of [19], the rank of the rigidity matrix is projectively invariant and hence f is an

invariant as in Section 2 of [26]. The first fundamental theorem of invariant theory [8] shows then that f can be expressed as a bracket algebra polynomial g . The vanishing of g is equivalent to the vanishing of a Cayley algebra expression (see Theorem 1 of [24]) and this proves the theorem. \square

The “inverse problem” is still open:

Problem 2. Suppose we are given by a certain system of geometric conditions. How to verify if this system defines some (G, k) -strata? How to find all the graphs G corresponding to these strata?

Finally, we formulate a problem for non-planar tensegrities.

Problem 3. Find analogous elementary geometric conditions in the three- (higher-) dimensional case.

We refer to [26] for examples of geometric conditions in dimension 3.

5.4. Complexity of the strata. We end this section with a discussion of the complexity of geometric (G, k) -strata.

A geometric (G, k) -stratum is defined by some union of the conditional solutions of systems of geometric conditions. Each system in this union has its own conditional number. Take the maximal among all the conditional numbers in the union. We call the minimal number among such maximal numbers for all the unions of systems defining the same (G, k) -stratum the *geometric complexity* of the (G, k) -stratum.

Example 5.6. The geometric complexity of $(G_0, 1)$ stratum for the graph G_0 described in Subsection 5.1 and shown in Figure 2 on the left equals 3.

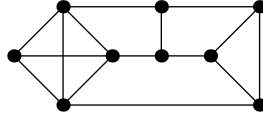
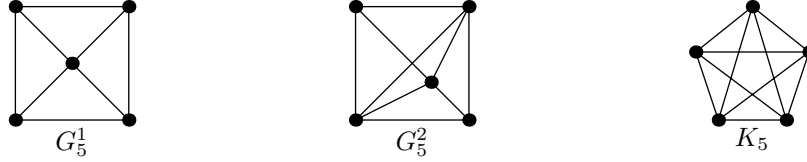
Problem 4. Find the asymptotics of the maximal complexity of geometric (G, k) -strata with bounded number of vertices while k tends to infinity.

6. PLANE TENSEGRITIES WITH A SMALL NUMBER OF VERTICES

In this section we work in the two-dimensional case (unless otherwise stated). In Subsection 6.1 we study the 2-TC of graphs. In particular, we calculate the 2-TC for sufficiently connected graphs with seven or less vertices. In Subsection 6.2 we give a list of geometric conditions for realizability of tensegrities in the plane for graphs with zero 2-TC.

6.1. On the tensegrity 2-characteristic of graphs. How to characterize the graphs that have a certain number of vertices, edges, and a given 2-TC? Corollary 2.6 of Laman’s theorem gives us an answer for graphs $G = (V, E)$ with $2|V| - |E| = 3$ and $\tau_2(G) = 0$: they are isostatic. For other cases not much is known. In this subsection we have a first glance at the general problem.

Recall the following definitions from graph theory. Let G be a graph. The *vertex connectivity* $\kappa(G)$ is the minimal number of vertices whose deletion disconnects G . The *edge connectivity* $\lambda(G)$ is the minimal number of edges whose deletion disconnects G . It is well known that $\kappa(G) \leq \lambda(G)$.


 FIGURE 7. A graph G with 9 vertices, 15 edges and $\tau_2(G) = 1$.

 FIGURE 8. The three possible graphs with five vertices, $\kappa \geq 2$ and $\lambda \geq 3$.

For general dimension d , let $G(P)$ be a framework in \mathbb{R}^d with underlying graph G . If $\kappa(G) < d$ or $\lambda(G) < d + 1$ then $G(P)$ consists of two or more pieces that can rotate with respect to each other. So for us the most interesting graphs are those with $\kappa(G) \geq d$ and $\lambda(G) \geq d + 1$.

Proposition-Example 6.1. Let G be a 2-vertex and 3-edge connected graph with k edges and n vertices. If $n \leq 7$, then

$$\tau_2(G) = k - 2n + 3.$$

Remark 6.2. In particular we have equality in Corollary 3.13 under the conditions of Proposition-Example 6.1. The formula of Proposition-Example 6.1 holds for many graphs in general, see for instance Example 3.15. It does not always hold for graphs with 9 vertices as the example below shows.

Example 6.3. Let G be the graph with 9 vertices and 15 edges as in Figure 7. The graph G satisfies $k = 2n - 3$ but it is not isostatic, so $\tau_2(G) > 0$. In fact it is not hard to prove that $\tau_2(G) = 1$. So the formula of Proposition-Example 6.1 does not hold for all graphs. Note that G contains the complete graph K_4 as an induced subgraph, so this implies that the stresses on the edges that do not belong to this K_4 are zero for a framework in a codimension 0 stratum of $B_2(G)$.

Proof of Proposition-Example 6.1. We use a classification argument.

Four vertices. For the complete graph K_4 we have $\tau_2(K_4) = 1 = 6 - 8 + 3$. There are no other graphs satisfying the conditions of the proposition.

Five vertices. There are three possibilities, we show them in Figure 8. From Proposition 3.10 we know that $\tau_2(K_5) = 3 = 10 - 10 + 3$ and in Example 4.5 we have seen that $\tau_2(G_5^1) = 1 = 8 - 10 + 3$. To see that $\tau_2(G_5^2) = 2$ we apply Corollary 3.14.

Six vertices. From the classification of graphs on six vertices (see for instance [23]) we know that any such 2-vertex and 3-edge connected graph can be obtained by adding

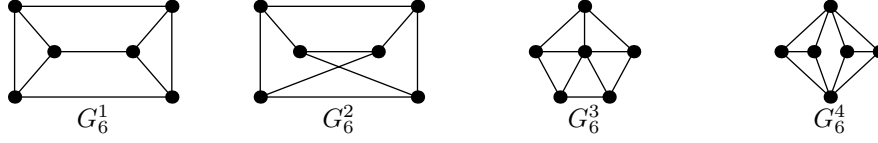


FIGURE 9. The four graphs with six vertices, $\kappa \geq 2$, $\lambda \geq 3$ and a minimal number of edges.

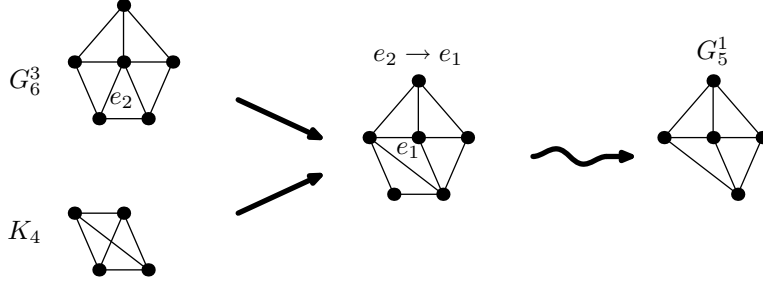


FIGURE 10. Using Proposition 4.1 we get that $\tau_2(G_6^3) = \tau_2(G_5^1) = 1$.

edges to one of the four graphs shown in Figure 9. By Corollary 3.14 it suffices to check the formula of the proposition for them.

Note that G_6^1 and G_6^2 are isostatic, so they both have zero 2-TC. For G_6^3 we proceed as follows. From Corollary 3.13 it follows that $\tau_2(G_6^3) \geq 1$. Then we use Proposition 4.1 in the same way as in Example 4.5 to show that

$$\tau_2(G_6^3) = \tau_2(G_5^1) = 1, \quad \text{and again} \quad 10 - 12 + 3 = 1,$$

see Figure 10.

It is easy to see that the same argument works to show that $\tau_2(G_6^4) = 1$.

Seven vertices. From the classification of graphs with seven vertices (see [23]) we get that all 2-vertex and 3-edge connected graphs on seven vertices can be obtained by adding edges to one of the seven graphs shown in Figure 11. By Corollary 3.14 it suffices again to check these graphs.

The graphs G_7^1 , G_7^2 , G_7^3 , and G_7^4 are isostatic, so they all have zero 2-TC.

The other three have 12 edges. We apply Corollary 4.8 to G_6^3 and G_6^4 to obtain that

$$\tau_2(G_7^5) = 1 \quad \text{and} \quad \tau_2(G_7^7) = 1.$$

To prove that the 2-TC of $G_7^6 = K_{3,4}$ is 1 we proceed as follows. First, $\tau_2(G_7^6) \geq 1$ by Corollary 3.13. Then we apply Proposition 4.1 as shown in Figure 12. The graph G has 6 vertices and 10 edges and thus we have $\tau_2(G) = 1$. It is easy to check that for a general position framework $G(P)$ with a nonzero self stress, all edges of $G(P)$ have nonzero stress. On the middle picture we get a vertex of degree 2, so we reduce to the graph H on the right. Note that H is isomorphic to G , so $\tau_2(H) = 1$. Hence $\tau_2(G_7^6) = 1$ as well.

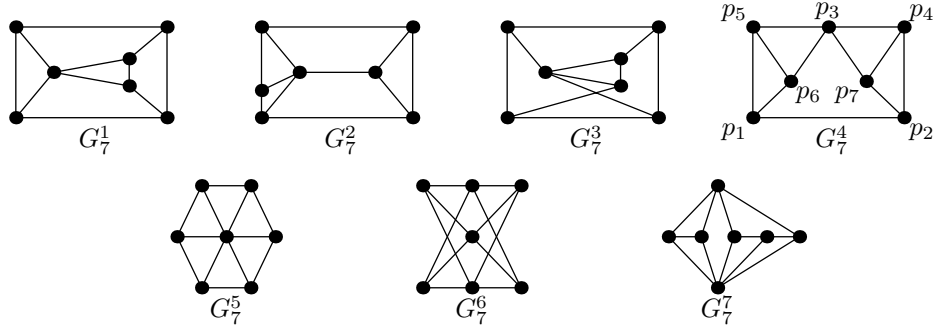


FIGURE 11. The seven graphs with seven vertices, $\kappa \geq 2$, $\lambda \geq 3$ and a minimal number of edges.

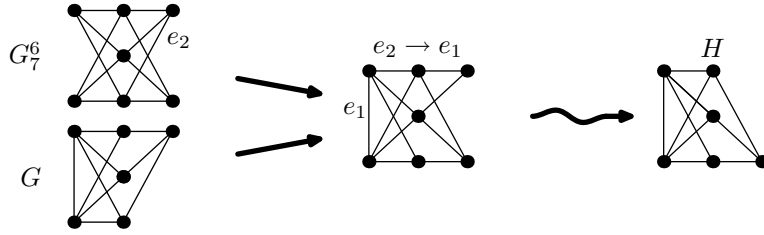


FIGURE 12. Using Proposition 4.1 one sees that $\tau_2(G_7^6) = 1$.

□

6.2. Geometric conditions for realizability of plane tensegrities for graphs with zero tensegrity 2-characteristic. Like in intersection theory of algebraic varieties, it often happens that strata for a graph with negative 2-TC are obtained as intersections of closures of some strata of graphs with zero 2-TC. So the conditions for realizability of plane tensegrities for graphs with zero 2-TC are the most important. In this subsection we give all the conditions for the zero 2-TC graphs with number of vertices not exceeding 8.

In practice one would like to construct a tensegrity without struts or cables with zero edge-stress. So it is natural to give the following definition. We say that a graph G is *visible* at the framework P if there exists a self stress that is nonzero at each edge of this framework.

Remark 6.4. Visibility restrictions remove many degenerate strata. For instance if a zero 2-TC graph G has a complete subgraph on vertices v_1 , v_2 , and v_3 , then the codimension 1 stratum defined by the condition: *the points v_1 , v_2 , and v_3 are on one line* does in many cases not contain visible frameworks.

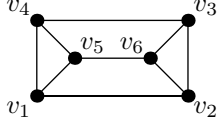
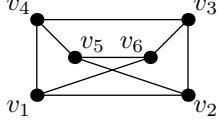
Let us list the geometric conditions for the vertices of all visible 2-vertex and 3-edge connected graphs with n vertices and zero 2-TC for $n \leq 8$. To find the geometric conditions we essentially use the surgeries of Section 4, see Propositions 4.1, 4.7 and 4.9.

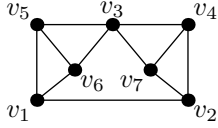
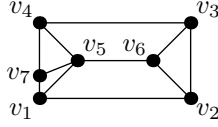
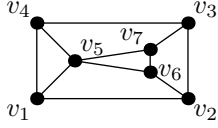
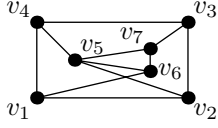
In the next table we use besides the elementary also the following two additional geometric conditions:

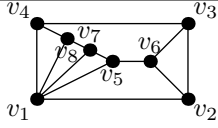
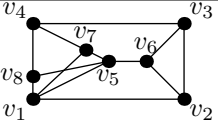
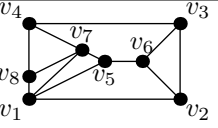
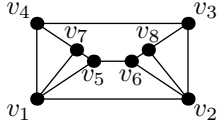
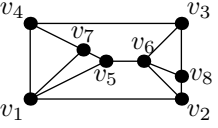
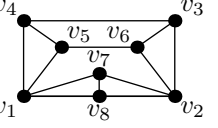
- six points are on a conic;
- for six points p_1, \dots, p_6 the lines p_1p_2 , p_3p_4 , and p_5p_6 have a common nonempty intersection.

As we have seen in Examples 5.2 and 5.3 these conditions are equivalent to geometric conditional systems.

Geometric conditions for some of the graphs in the table were known before. For instance, geometric conditions in terms of bracket polynomials for both 6-vertex graphs in the table, for the third 7-vertex graph, and the twenty-seventh 8-valent graph are given in [26] by N. L. White and W. Whiteley. For the relation of bipartite graphs (the second 6-vertex graph here) with rectangular bar constructions see the paper [4] by E. D. Bolker and H. Crapo.

| Graph (6 vert.) | Sufficient geometric conditions |
|---|--|
|  | the lines v_1v_2 , v_3v_4 , and v_5v_6 have a common nonempty intersection |
|  | the six points v_1, v_2, v_3, v_4, v_5 , and v_6 are on a conic |

| Graph (7 vert.) | Sufficient geometric conditions |
|---|---|
|  | $v_1 \nabla v_2 \nabla v_3 = 0$ |
|  | the lines v_1v_2 , v_3v_4 , and v_5v_6 have a common nonempty intersection |
|  | the lines v_1v_2 , v_3v_4 , and v_5p where $p = [v_2, v_6; v_3, v_7]$ have a common nonempty intersection |
|  | the six points v_1, v_2, v_3, v_4, v_5 , and p , where $p = [v_1, v_6; v_3, v_7]$ are on a conic |

| Graph (8 vert.) | Geometric conditions |
|---|--|
|    | the lines v_1v_2 , v_3v_4 , and v_5v_6 have a common nonempty intersection |
|    | $v_1 \nabla v_2 \nabla v_3 = 0$ |

| Graph (8 vert.) | Geometric conditions |
|-----------------|---|
| | the six points v_1, v_2, v_3, v_4, v_5 , and v_6 are on a conic |
| | the lines v_1v_2, v_3v_4 , and v_5p , where $p = [v_2, v_6; v_3, v_7]$ have a common nonempty intersection |
| | the lines v_1v_2, v_3v_4 , and v_5p , where $p = [v_2, v_6; v_7, v_8]$ have a common nonempty intersection |
| | the six points v_1, v_2, v_3, v_4, v_5 , and p , where $p = [v_1, v_6; v_3, v_7]$, are on a conic |
| | the lines v_1v_2, v_3p , and v_5q , where $p = [v_1, v_4; v_5, v_8]$ and $q = [v_2, v_6; v_3, v_7]$ have a common nonempty intersection |
| | the lines v_5v_6, v_1p , and v_4q , where $p = [v_2, v_3; v_6, v_7]$ and $q = [v_2, v_3; v_6, v_8]$ have a common nonempty intersection |
| | the six points v_1, v_2, v_4, v_6, p , and q , where $p = [v_2, v_3; v_6, v_7]$ and $q = [v_2, v_5; v_6, v_8]$, are on a conic |
| | the six points v_1, v_3, v_4, v_6, p , and q , where $p = [v_2, v_3; v_5, v_7]$ and $q = [v_5, v_7; v_6, v_8]$, are on a conic |
| | the six points v_1, v_2, v_3, v_5, p , and q , where $p = [v_1, v_6; v_3, v_7]$ and $q = [v_3, v_4; v_5, v_8]$, are on a conic |
| | the six points v_1, v_2, v_3, v_5, v_6 , and q , where $p = [v_1, q; v_3, v_4]$ and $q = [v_5, v_7; v_4, v_8]$, are on a conic |

| Graph (8 vert.) | Geometric conditions |
|-----------------|---|
| | the six points v_1, v_2, v_4, v_5, p , and q , where $p = [v_1, v_6; v_5, v_8]$ and $q = [p, v_7; v_2, v_3]$, are on a conic |
| | the three points $[v_1, v_4; v_2, v_3]$, $[v_1, v_5; v_2, v_6]$, and $[v_5, v_8; v_6, v_7]$ are on one line |

| Graph (8 vert.) | Sufficient geometric conditions |
|-----------------|--|
| | the three points $[v_1, v_2; v_6, v_7]$, $[v_1, p; v_6, v_8]$, and $[p, q; v_3, v_8]$, where $p = [v_2, v_4; v_5, v_8]$ and $q = [v_1, v_5; v_3, v_4]$, are on one line, AND the lines $p'v_2$, $q'v_3$, and v_6v_7 have a common nonempty intersection, where $p' = [r', s'; v_1, v_6]$, $q' = [r', s'; v_6, v_8]$, $r' = [v_1, v_4; v_2, v_5]$, and $s' = [v_3, v_4; v_5, v_8]$ |
| | the six points v_1, v_4, v_7, v_8, p , and q , where $p = [r, s; v_3, v_4]$, $q = [r, s; v_5, v_8]$, $r = [v_1, v_2; v_5, v_6]$, and $s = [v_2, v_3; v_6, v_7]$, are on a conic, AND the six points v_1, v_2, v_6, v_7, p' , and q' , where $p' = [r', s'; v_2, v_3]$, $q' = [r', s'; v_5, v_6]$, $r' = [v_1, v_4; v_5, v_8]$, and $s' = [v_3, v_4; v_7, v_8]$, are on a conic |
| | the six points v_1, v_3, v_4, v_5, v_7 , and p , where $p = [v_1, q; v_7, v_8]$, $q = [r, s; v_2, v_3]$, $r = [v_3, v_6; v_7, v_8]$, and $s = [v_1, v_6; v_2, v_8]$, are on a conic, AND the six points v_1, v_2, v_3, v_6, v_8 , and p' , where $p' = [v_3, q'; v_7, v_8]$, $q' = [r', s'; v_1, v_4]$, $r' = [v_1, v_5; v_7, v_8]$, and $s' = [v_3, v_5; v_4, v_7]$, are on a conic |

Remark 6.5. For the last three graphs in the table we have two distinct equations. Nevertheless, the 2-TC of the graphs are zero. This is similar to the case of non-complete intersections in algebraic geometry.

REFERENCES

- [1] M. Barnabei, A. Brini, G.-C. Rota, *On the exterior calculus of invariant theory*, J. Algebra, vol. 96, n. 1 (1985), pp. 120–160.
- [2] A. R. Berg, T. Jordán, *A proof of Connelly's conjecture on 3-connected circuits of the rigidity matroid*, J. Combin. Theory Ser. B vol. 88, n. 1 (2003), pp. 77–97.
- [3] J. Bochnak, M. Coste, M.-F. Roy, *Géométrie algébrique réelle*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Folge 3, vol 12, Springer, 1987.
- [4] E. D. Bolker and H. Crapo, *Bracing rectangular frameworks. I*, SIAM J. Appl. Math. vol. 36, n. 3 (1979), pp. 473–490.
- [5] D. L. D. Caspar and A. Klug, *Physical principles in the construction of regular viruses*, in Proceedings of Cold Spring Harbor Symposium on Quantitative Biology, vol. 27 (1962), pp. 1–24.

- [6] R. Connelly, *Rigidity*, Chapter 1.7 of Handbook of convex geometry, vol. A. Edited by P. M. Gruber and J. M. Wills, North-Holland Publishing Co., Amsterdam (1993), pp. 223–271.
- [7] R. Connelly and W. Whiteley, *Second-order rigidity and prestress stability for tensegrity frameworks*, SIAM Journal of Discrete Mathematics, vol. 9, n. 3 (1996), pp. 453–491.
- [8] P. Doubilet, G.-C. Rota, J. Stein, *On the foundations of combinatorial theory. IX.*, Combinatorial methods in invariant theory. Studies in Appl. Math. vol. 53 (1974), pp. 185–216.
- [9] B. Jackson, T. Jordán, *Connected rigidity matroids and unique realizations of graphs*, J. Combin. Theory Ser. B, vol. 94, n. 1 (2005), pp. 1–29.
- [10] R. Haas, D. Orden, G. Rote, F. Santos, B. Servatius, H. Servatius, D. Souvaine, I. Streinu, W. Whiteley, *Planar minimally rigid graphs and pseudo-triangulations*, Comput. Geom., vol. 31, n. 1–2 (2005), pp. 31–61.
- [11] D. E. Ingber, *Cellular tensegrity: defining new rules of biological design that govern the cytoskeleton*, Journal of Cell Science, vol. 104 (1993), pp. 613–627.
- [12] M. de Guzmán, *Finding Tensegrity Forms*, preprint, 2004.
- [13] M. de Guzmán, D. Orden, *Finding tensegrity structures: Geometric and symbolic aproaches*, in Proceedings of EACA-2004, pp. 167–172, 2004.
- [14] M. de Guzmán, D. Orden, *From graphs to tensegrity structures: Geometric and symbolic approaches*, Publ. Mat. 50 (2006), pp. 279–299.
- [15] G. Laman, *On graphs and rigidity of plane skeletal structures*, J. Engrg. Math., vol. 4 (1970), pp. 331–340.
- [16] J. C. Maxwell, *On reciprocal figures and diagrams of forces*, Philos. Mag. vol. 4, n. 27 (1864), pp. 250–261.
- [17] R. Motro, *Tensegrity: Structural systems for the future*, Kogan Page Science, London, 2003.
- [18] D. Orden, G. Rote, F. Santos, B. Servatius, H. Servatius, W. Whiteley, *Non-crossing frameworks with non-crossing reciprocals*, Discrete Comput. Geom., vol. 32, n. 4 (2004), pp. 567–600.
- [19] B. Roth, W. Whiteley, *Tensegrity frameworks*, Trans. Amer. Math. Soc., vol. 265, n. 2 (1981), pp. 419–446.
- [20] B. Servatius, *Tensegrities*, PAMM, vol. 7, n. 1 (2007), pp. 1070101–1070102.
- [21] R. E. Skelton, *Deployable tendon-controlled structure*, United States Patent 5642590, July 1, 1997.
- [22] K. Snelson, <http://www.kennethsnelson.net>.
- [23] S. Speed, <http://www.msci.memphis.edu/~speeds/>.
- [24] B. Sturmfels, W. Whiteley, *On the synthetic factorization of projectively invariant polynomials*, J. Symbolic Computation, vol. 11 (1991), pp. 439–453.
- [25] A. G. Tibert, *Deployable tensegrity structures for space applications*, Ph.D. Thesis, Royal Institute of Technology, Stockholm 2002.
- [26] N. L. White, W. Whiteley, *The algebraic geometry of stresses in frameworks*, SIAM J. Alg. Disc. Meth., vol. 4, n. 4 (1983), pp. 481–511.
- [27] W. Whiteley, *Rigidity and scene analysis*, in J. E. Goodman and J. O’Rourke, editors, Handbook of Discrete and Computational Geometry, chapt. 49, pp. 893–916, CRC Press, New York, 1997.

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